Holstein-Primakoff Realizations on Coadjoint Orbits

Phillial Oh[*]

Department of Physics

Sung Kyun Kwan University

Suwon 440-746, Korea

Chaiho Rim[**]

Department of Physics

Chonbuk National University

Chonju, 561-756, Korea

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Abstract

We derive the Holstein-Primakoff oscillator realization on the coadjoint orbits of the $SU(N+1)$ and $SU(1,N)$ group by treating the coadjoint orbits as a constrained system and performing the symplectic reduction. By using the action-angle variables transformations, we transform the original variables into Darboux variables. The Holstein-Primakoff expressions emerge after quantization in a canonical manner with a suitable normal ordering. The corresponding Dyson realizations are also obtained and some related issues are discussed.

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I. INTRODUCTION

It is well known that the Holstein-Primakoff (HP) [1] and Dyson [2] realizations of $su(2)$ algebra in terms of a single oscillator are very useful in describing the spin-density wave phenomena and many others in condensed matter physics [3] and nuclear physics [4]. The HP realization also appears in the q-deformation [5] of the quantum algebras $su_q(2)$ and $su(1,1)_q$ [6] although q-deformation approach of of Jordan-Schwinger type [7,8] is more conventional.

Since the HP and Dyson representations of $su(2)$ algebra can be interpreted as quantum mechanical operators on S^2, which is the coadjoint orbit of $SU(2)$ group, it is useful to consider them on the coadjoint orbits of an arbitrary group in extending to higher group [9]. So far, the generalization was performed mostly to minimal $CP(N)$ orbits [10] or Grassmanian manifold [11] which was largely based on the coherent state method [12].

In this letter, we discuss general representations of HP and Dyson oscillator realizations for the $su(N+1)$ and $su(1,N)$ algebras on the coadjoint orbits of $SU(N+1)$ and $SU(1,N)$ by treating the coadjoint orbits as a constrained classical system and by explicitly performing a symplectic reduction. Compared with non-linear realization method on coset space [10], this approach can have some advantage of exploiting the well-developed mathematical tool of symplectic reduction [13] which in our case deals mainly with quadratic constraints. The HP realization will emerge, if we transform the reduced system into canonical one by using the action-angle variable and then quantize it in a standard manner with the normal ordering prescription. Then, the Dyson realization will be obtained by shifting the square-root factor in HP realizations [14]. One of the merits of this coadjoint orbit approach is to provide a unified framework for finding explicit expressions for HP and Dyson realization in the compact and non-compact case. We will be mainly concerned with minimal and maximal orbits of $SU(N+1)$ and $SU(1,N)$ to make the presentation simple.

We start by briefly explaining our notation. Let us denote a column vector as a ket $|Z> = (Z^{(1)}, Z^{(2)}, \cdots, Z^{(N+1)})^T$ and introduce its bra $< \bar{Z}| = (\bar{Z}_{(1)}, \bar{Z}_{(2)}, \cdots, \bar{Z}_{(N+1)})$. Then,

$< \bar{Z}|Z> = \sum_{i=1}^{N+1} \bar{Z}_{(i)} Z^{(i)} \equiv \bar{Z}_{(i)} M^{ij} Z_{(j)}$. The raising and lowering are done with respect to the metric $M = \text{diag}(1, \epsilon, \cdots, \epsilon)$. $\epsilon = 1(-1)$ for $SN(N+1)(SU(1,N))$. Let us express the element g of $SU(N)$ and $SU(1,N)$ by $N+1$ kets $(|Z_1>, |Z_2>, \cdots, |Z_{N+1}>)$ with $|Z_p> = (Z_p^{(1)}, Z_p^{(2)}, \cdots, Z_p^{(N+1)})^T$. Then, $Mg^\dagger M$ is composed of $N+1$ bras $< \bar{Z}^p|$'s such that $< \bar{Z}^p| = (\bar{Z}_{(1)}^p, \bar{Z}_{(2)}^p, \cdots, \bar{Z}_{(N+1)}^p)$. With the notation $< \bar{Z}_p| = < \bar{Z}^q|M_{qp}$, $Mg^\dagger Mg = I$ gives

$$< \bar{Z}_p|Z_q> = M_{pq}, \quad \det(|Z_1>, |Z_2>, \cdots, |Z_{N+1}>) = 1. \tag{1.1}$$

The isospin charges on the coadjoint orbits are defined by

$$Q^a = -2\text{Tr}(gxg^{-1}T^a). \tag{1.2}$$

where $x = i\,\text{diag}(x_1, x_2, \cdots, x_{N+1})$ with $\sum_{i=1}^{N+1} x_i = 0$. The x_i's are real and T^a's are the anti-hermitian generators of the group which satisfy the Lie algebra with real structure constant $f_{ab}{}^c$: $[T_a, T_b] = f_{ab}{}^c T_c$ and $\text{Tr}(T^a T^b) = -\frac{1}{2}\eta^{ab}$. By making use of the second equation of the Eq.(1.1), $|Z_{N+1}>$ can be eliminated and subsequently we find that Q^a can be expressed as [15]

$$Q^a = -2i \sum_{p=1}^{N} J_p < \bar{Z}^p|T^a|Z_p> . \tag{1.3}$$

where $J_p = x_p - x_{N+1} = x_1 + \cdots + 2x_p + \cdots + x_N$.

Let us consider a classical system defined on the coadjoint orbit of $SU(N)$ described by a Lagrangian

$$L = 2\text{Tr}(xg^{-1}\dot{g}) - H(Q^a) = 2i \sum_{p=1}^{N+1} x_p < \bar{Z}^p|\frac{d}{dt}|Z_p> -H(Q^a). \tag{1.4}$$

By using the second equation of the Eq.(1.1) again, we find

$$L = 2i \sum_{p=1}^{N} J_p < \bar{Z}^p|\frac{d}{dt}|Z_p> -H(Q^a). \tag{1.5}$$

Note that there still exist the constraints $< \bar{Z}_p|Z_q> -M_{pq} = 0$ $(p,q = 1, \cdots, N)$.

Using the symplectic structure of the above Lagrangian, one can show that the isospin charges satisfy the $su(N+1)$ and $su(1,N)$ algebras [15]:

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$$\{Q^a, Q^b\} = f^{ab}{}_c Q^c. \tag{1.6}$$

HP realizations will be found if one finds a quantum mechanical expression of the above isospin charges in terms of canonical variables and so it is essential to bring the Lagrangian (1.4) into a canonical form. We will achieve this by transforming the above system into action-angle variables. In passing, we mention that action-angle variables approach on the coadjoint orbits was also considered before [16] in the path integral quantization of the orbits in the compact case.

II. MINIMAL ORBITS

Let us first apply the above formalism to minimal orbits, $CP(N)$ and its non-compact counter part. In this case, we have $x = i \operatorname{diag}(J, -J/N, \cdots, -J/N)$ and $J_1 = J$, $J_2 = \cdots = J_N = 0$. In the compact case, J is an integer for quantizable orbits. For non-compact case, J depends on the various types of representations of non-compact groups [17]. With the notation $Z = (Z_0, Z_1, \cdots, Z_N)^T$ and introducing $\bar{Z} = (\bar{Z}_0, \bar{Z}_1, \cdots, \bar{Z}_N)$, we find that the Lagrangian can be written as

$$L_Z = iJ(\bar{Z} M \dot{Z} - \dot{\bar{Z}} M Z) - H(Q^a) \tag{2.1}$$

with the constraint $\bar{Z} M Z = 1$. Note that the notation in the above equation denotes the conventional matrix product rather that the abstract bracket inner product. In addition, the component is relabeled from 0 to N instead of 1 to $N + 1$. We mention that the above Lagrangian in the compact case was used in describing the internal degrees of freedom of non-Abelian Chern-Simons particles [18].

It is well known that the constraint can be solved explicitly in terms of the projective coordinates defined by $\xi_i = Z_i/Z_0 (Z_0 \neq 0, i = 1, 2, \cdots, N)$ with a real gauge condition: [16,18,19]

$$\chi = \frac{1}{2}(Z_0^* - Z_0) = 0. \tag{2.2}$$

Then, the solution to the constraint $\bar{Z}MZ = 1$ is given by

$$\bar{Z}_0 = Z_0 = \frac{1}{\sqrt{1 + \epsilon|\xi|^2}}, \qquad |\xi|^2 = \sum_i |\xi_i|^2, \tag{2.3}$$

By substituting $Z_I = (Z_0, Z_0\xi_i)$ and Eq. (2.3) into Eq. (2.1), we obtain the following reduced Lagrangian:

$$L_\xi = iJ\epsilon\frac{\bar{\xi}\dot{\xi} - \dot{\bar{\xi}}\xi}{1 + \epsilon \mid \xi \mid^2} - H(Q^a). \tag{2.4}$$

The isospin charges of Eq. (1.3) becomes

$$Q^a = -i\frac{2J}{1 + \epsilon|\xi|^2}\left(T_{00}^a + T_{0i}^a\xi_i + \epsilon T_{i0}^a\bar{\xi}_i + \epsilon T_{ij}^a\bar{\xi}_i\xi_j\right). \tag{2.5}$$

To make contact with HP representations, we make the following action-angle transformation of variables [20]:

$$I_i = \frac{2|J||\xi_i|^2}{1 + \epsilon|\xi|^2} \equiv \bar{\alpha}_i\alpha_i, \tag{2.6}$$

and the angle variables are given by the phases of the α_i's. Assuming a positive value for $J\epsilon$, we have the Lagrangian (2.4) given by

$$L_\alpha = \frac{i}{2}(\bar{\alpha}_i\dot{\alpha}_i - \dot{\bar{\alpha}}_i\alpha_i) - H(Q^a). \tag{2.7}$$

The Poisson bracket is defined in a canonical way

$$\{\alpha_i, \bar{\alpha}_j\} = i\delta_{ij}. \tag{2.8}$$

Note that J is negative in the non-compact case. Otherwise, the role of α_i and $\bar{\alpha}_i$ would be interchanged in the canonical commutation relation (2.8). The isospin functions (2.5) are expressed as follows:

$$Q^a = -i\left[(2J - \epsilon|\alpha|^2)T_{00}^a + \epsilon\bar{\alpha}_iT_{ij}^a\alpha_j \right.$$
$$\left. + \ \epsilon\sqrt{2J - \epsilon|\alpha|^2}\bar{\alpha}_iT_{i0}^a + \sqrt{2J - \epsilon|\alpha|^2}T_{0i}^a\alpha_i\right]. \tag{2.9}$$

The quantum mechanical operator realizations are obtained after quantizing the above operators by replacing the Poisson bracket (2.8) with Dirac bracket $\alpha_i \longrightarrow a_i^\dagger$, $\bar{\alpha}_i \longrightarrow a_i$, and

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perform the normal ordering of the resulting operators by putting the creation operators $a^\dagger$ to the left of annihilation operators a. Following the above procedure, we get the following HP realization:

$$\hat{Q}^a_{\text{hp}} = -i\left[(2J - \epsilon a^\dagger \cdot a)T^a_{00} + \epsilon T^a_{ij}a^\dagger_j a_i \right.$$
$$\left. + \epsilon\sqrt{2J - \epsilon a^\dagger \cdot a}\, T^a_{i0}a_i + a^\dagger_i T^a_{0i}\sqrt{2J - \epsilon a^\dagger \cdot a}\right]. \tag{2.10}$$

If we shift the square root in front of $T^a_{i0}a_i$ to the back of $a^\dagger_i T^a_{0i}$, we get the following generalized Dyson realization:

$$\hat{Q}^a_{\text{d}} = -i\left\{\epsilon[T^a_{i0} + T^a_{ij}a^\dagger_j - T^a_{00}a^\dagger_i - T^a_{0j}a^\dagger_j a^\dagger_i]a_i \right.$$
$$\left. + 2JT^a_{00} + 2JT^a_{0i}a^\dagger_i\right\}. \tag{2.11}$$

It is easy to check that the above realization satisfies the algebras for both the compact and non-compact cases. For the compact case with $\epsilon = 1$, the above expression was obtained as an holomorphic differential operator acting on coherent state [15]. Note that shifting the square root makes $\hat{Q}^a$'s and $\hat{Q}^{a\dagger}$'s not manifestly conjugate to each other in the Dyson case. To make the representation unitary, the inner product should be defined [14] with respect to the Liouville measure, while the Bargmann measure is used for the HP case.

We note that a similar expression in the compact case appeared in the study of the generalized spin system [10]. Our result reduces to it after a trivial rescaling of the variables and choosing a specific representation. Our phase space is in the canonical form (see the Eq. (2.8)) and the result holds for arbitrary representation of the group. In addition, Eq. (2.10) also covers the non-compact case.

To put the above expressions into a more familiar form, we consider $SU(N+1)$ case. It is convenient to use a representation in which the the ladder operators, $E^a_{pq} \equiv E^{\alpha\beta}_{pq}$ ($\alpha, \beta = 1, \cdots, N; p, q = 0, 1, \cdots N$), are given as follows:

$$E^{0\alpha}_{pq} = -\delta^0_p \delta^\alpha_q, \quad E^{\alpha 0}_{pq} = -\delta^\alpha_p \delta^0_q, \quad E^{\alpha\beta}_{pq} = \delta^\alpha_p \delta^\beta_q \ (p \neq q), \tag{2.12}$$

and the Cartan subalgebra is given by N diagonal matrices which are denoted by $H^m \equiv T^{m^2+2m}$, $m = 1, 2, \cdots N$ and expressed as follows:

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$$H_{pq}^m = (\sum_{k=0}^{m-1} \delta_{pk}\delta_{qk} - m\delta_{pm}\delta_{qm})/\sqrt{2m(m+1)}. \tag{2.13}$$

Then, we obtain the following HP realization from the Eq. (2.10):

$$\hat{Q}^{0i} = a_i^\dagger \sqrt{2J - a^\dagger \cdot a}, \quad \hat{Q}^{i0} = \sqrt{2J - a^\dagger \cdot a}\, a_i = \hat{Q}^{0i\dagger}$$

$$\hat{Q}^{ij} = -a_j^\dagger a_i \quad (i \neq j) \tag{2.14}$$

$$\hat{Q}^m = \frac{1}{\sqrt{2m(m+1)}}\left((m+1)a_m^\dagger a_m + \sum_{k=m+1}^N a_k^\dagger a_k - 2J\right).$$

The corresponding Dyson realizations of $SU(N+1)$ is given by [21]:

$$\hat{Q}^{0i} = a_i^\dagger(2J - a^\dagger \cdot a), \quad \hat{Q}^{i0} = a_i$$

$$\hat{Q}^{ij} = -a_j^\dagger a_i \quad (i \neq j) \tag{2.15}$$

$$\hat{Q}^m = \frac{1}{\sqrt{2m(m+1)}}\left((m+1)a_m^\dagger a_m + \sum_{k=m+1}^N a_k^\dagger a_k - 2J\right).$$

III. MAXIMAL ORBITS

Now, let us turn to the maximal orbits, flag manifold of the group. Here, in order to make the presentation simple, we will restrict to the $SU(3)$ and $SU(1,2)$ case. Extension to higher group is straightforward. Let us choose the element x as $x = i\,\mathrm{diag}(x_1, x_2, -(x_1+x_2))$. Then, $J_1 = 2x_1 + x_2, J_2 = x_1 + 2x_2$. We require $x_1 \neq x_2, J_1 \neq 0, J_2 \neq 0$. Introduce again $Z_i = (Z_{i0}, Z_{i1}, Z_{i2})^T$ $(i = 1, 2)$ and $\bar{Z}_i = (\bar{Z}_{i0}, \bar{Z}_{i1}, \bar{Z}_{i2})$, we find

$$L = i\sum_{i=1,2} J_i M^{ii}(\bar{Z}_i M\dot{Z}_i - \dot{\bar{Z}}_i M Z_i) - H(Q^a). \tag{3.1}$$

The constraints are given by

$$\bar{Z}_i M Z_j = M_{ij} \tag{3.2}$$

To solve the constraints, we again choose the real gauge conditions:

$$\bar{Z}_{10} = Z_{10} \ (\neq 0), \quad \bar{Z}_{22} = Z_{22} \ (\neq 0). \tag{3.3}$$

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Defining the projective coordinates $z_i = Z_{1i}/Z_{10}$, $\xi_\alpha = Z_{2\alpha}/Z_{20}$ ($i = 1, 2; \alpha = 0, 1$), the above constraints can be solved as

$$Z_{10} = \frac{1}{\sqrt{1 + \epsilon|z|^2}}, \quad Z_{22} = \frac{1}{\sqrt{1 + \epsilon|\xi_0|^2 + |\xi_1|^2}}, \tag{3.4}$$

and the remaining constraints become $\xi_0 = -\epsilon(\xi_1 \bar{z}_1 + \bar{z}_2)$. To compare with the known results of symplectic structure on the maximal orbit [22], we redefine the variables by $\bar{\xi}_1 \to -z_3, \bar{\xi}_0 \to -z_4$ with the remaining constraint given by $z_4 = \epsilon(z_2 - z_1 z_3)$. Then the canonical one form of the Lagrangian (3.1) is given by $\theta = i(\partial - \bar{\partial})W$, where W is given by

$$W = \log(1 + \epsilon|z_1|^2 + \epsilon|z_2|^2)^m (1 + |z_3|^2 + \epsilon|z_2 - z_1 z_3|^2)^n \tag{3.5}$$

with $J_1 = m$ and $J_2 = -n$. In the compact case with $\epsilon = 1$ and m, n = integers, the above expression precisely reduces to the form given in Ref. [22]. For non-compact case, they need not be integers. From here on, we will use interchangeably use the variables z_1, z_2, z_3, z_4 or $z_1, z_2, w_0 \equiv z_4, w_1 \equiv z_3$. With our new notation, the isospin function Q^a's of the Eq.(1.3) becomes

$$Q^a = -i\frac{2m}{1 + \epsilon|z|^2}\left(T_{00}^a + T_{0i}^a z_i + \epsilon T_{i0}^a \bar{z}_i + \epsilon T_{ij}^a \bar{z}_i z_j\right)$$
$$- i\frac{2n}{1 + \epsilon|w_0|^2 + |w_1|^2}\left(-\epsilon T_{22}^a + \epsilon T_{2\alpha}^a \bar{w}_\alpha + T_{\alpha2}^a \zeta_\alpha - T_{\alpha\beta}^a \bar{w}_\beta \zeta_\alpha\right) \tag{3.6}$$

with $\zeta_\alpha = (w_0, \epsilon w_1)$.

Let us again consider the action-angle variable transformations

$$\alpha_i = \sqrt{2m}\frac{z_i}{\sqrt{1 + \epsilon|z|^2}}, \quad \beta_\alpha = \sqrt{2n}\frac{w_\alpha}{\sqrt{1 + \epsilon|w_0|^2 + |w_1|^2}} \tag{3.7}$$

which renders the Lagrangian (3.1) into a canonical form

$$L = \frac{i}{2}\left[\epsilon(\bar{\alpha}\dot{\alpha} - \dot{\bar{\alpha}}\alpha) + \epsilon(\bar{\beta}m\dot{\beta} - \dot{\bar{\beta}}m\beta)\right] - H(Q^a) \tag{3.8}$$

with m given by $m = \text{diag}(1, \epsilon)$. We also have the isospin functions given by

$$Q^a = -i\left[(2m - \epsilon|\alpha|^2)T_{00}^a + \epsilon\bar{\alpha}_i T_{ij}^a \alpha_j + \epsilon\sqrt{2m - \epsilon|\alpha|^2}\bar{\alpha}_i T_{i0}^a + \sqrt{2m - \epsilon|\alpha|^2}T_{0i}^a \alpha_i\right]$$
$$- i\left[-\epsilon(2n - \epsilon|\gamma|^2)T_{22}^a - \gamma_\alpha T_{\alpha\beta}^a \bar{\beta}_\beta + \sqrt{2n - \epsilon|\gamma|^2}\gamma_\alpha T_{\alpha2}^a + \epsilon\sqrt{2n - \epsilon|\gamma|^2}T_{2\alpha}^a \bar{\beta}_\alpha\right]. \tag{3.9}$$

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with $\gamma_\alpha = (\beta_0, \epsilon\beta_1)$. The quantum mechanical operator realizations are obtained after going through the same steps as in the minimal case.

To deal with the remaining constraint $z_4 = z_2 - z_1z_3$, we will restrict to the compact case for convenience. One is tempted to substitute this constraint directly into the Eq.(3.9) and then quantize the system. However, this would change the canonical structure of the Eq.(3.8) in a very complicated manner. Another way to carry out the analysis is to impose the constraint on the quantum state. The constraint in terms of $\alpha_1, \alpha_2, \alpha_3 \equiv \beta_1, \alpha_4 \equiv \beta_0$ is given by

$$\Phi_{\rm hp} = \alpha_4\sqrt{l_1} - \alpha_2\sqrt{l_2} + \alpha_1\alpha_3 = 0 \tag{3.10}$$

where $l_1 = 2m - |\alpha_1|^2 - |\alpha_2|^2$, $l_2 = 2n - |\alpha_3|^2 - |\alpha_4|^2$. One can easily check that the constraints are second class.

Using the expression (3.9) and canonically quantizing the system, we obtain the following HP realizations in the standard notation of the generators E's, the Eqs. (2.12) and (2.13):

$$\hat{Q}^{1+i2} = a_1^\dagger\sqrt{\hat{l}_1} - a_4^\dagger a_3, \ \hat{Q}^{1-i2} = \sqrt{\hat{l}_1}a_1 - a_3^\dagger a_4$$

$$\hat{Q}^3 = a_1^\dagger a_1 + \frac{1}{2}(a_2^\dagger a_2 - a_3^\dagger a_3 + a_4^\dagger a_4) - m$$

$$\hat{Q}^{4+i5} = a_2^\dagger\sqrt{\hat{l}_1} + a_4^\dagger\sqrt{\hat{l}_2}, \ \hat{Q}^{4-i5} = \sqrt{\hat{l}_1}a_2 + \sqrt{\hat{l}_2}a_4 \tag{3.11}$$

$$\hat{Q}^{6+i7} = -a_2^\dagger a_1 - a_3^\dagger\sqrt{\hat{l}_2}, \ \hat{Q}^{6-i7} = -a_1^\dagger a_2 - \sqrt{\hat{l}_2}a_3$$

$$\hat{Q}^8 = \frac{\sqrt{3}}{2}(a_2^\dagger a_2 + a_3^\dagger a_3 + a_4^\dagger a_4) - \frac{1}{\sqrt{3}}m - \frac{2}{\sqrt{3}}n$$

where $\hat{l}_1 = 2m - a_1^\dagger a_1 - a_2^\dagger a_2$, $\hat{l}_2 = 2n - a_3^\dagger a_3 - a_4^\dagger a_4$. Since the constraints are second class, only half of the constraints is imposed on the physical state

$$\hat{\Phi}_{\rm hp}|{\rm phys}> = (a_4^\dagger\sqrt{\hat{l}_1} - a_2^\dagger\sqrt{\hat{l}_2} + a_1^\dagger a_3^\dagger)|{\rm phys}> = 0. \tag{3.12}$$

The physical states are labeled by (m, n) and can be obtained by successive applications of m-times of $a_1^\dagger, a_2^\dagger$ combined and n-times of $a_3^\dagger, a_4^\dagger$ combined to the vacuum state. The above condition (3.12) will give some restrictions on m and n. The result will determine irreducible representations of the $SU(3)$ group according to the Borel-Weil-Bott theorem

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[23]. The detailed analysis on the relations between the Eqs. (3.11) and (3.12) and the irreducible representations is not of concern here and will be reported elsewhere.

By shifting the square root, we again get the corresponding constrained Dyson realizations:

$$\hat{Q}^{1+i2} = a_1^\dagger \hat{l}_1 - a_4^\dagger a_3, \ \hat{Q}^{1-i2} = a_1 - a_3^\dagger a_4$$

$$\hat{Q}^3 = a_1^\dagger a_1 + \frac{1}{2}(a_2^\dagger a_2 - a_3^\dagger a_3 + a_4^\dagger a_4) - m$$

$$\hat{Q}^{4+i5} = a_2^\dagger \hat{l}_1 + a_4^\dagger \hat{l}_2, \ \hat{Q}^{4-i5} = a_2 + a_4 \tag{3.13}$$

$$\hat{Q}^{6+i7} = -a_2^\dagger a_1 - a_3^\dagger \hat{l}_2, \ \hat{Q}^{6-i7} = -a_1^\dagger a_2 - a_3$$

$$\hat{Q}^8 = \frac{\sqrt{3}}{2}(a_2^\dagger a_2 + a_3^\dagger a_3 + a_4^\dagger a_4) - \frac{1}{\sqrt{3}}m - \frac{2}{\sqrt{3}}n$$

And we can infer that the constraint changes into

$$\hat{\Phi}_d|\text{phys} >= (a_4^\dagger - a_2^\dagger + a_1^\dagger a_3^\dagger)|\text{phys} >= 0. \tag{3.14}$$

Let us compare the above formula with the other Dyson realization which can be obtained by the method of geometric quantization in the holomorphic coherent state approach [24]:

$$\hat{Q}^1 = -\frac{1}{2}\left[(z_1^2 - 1)\frac{\partial}{\partial z_1} + z_1 z_2 \frac{\partial}{\partial z_2} + (z_2 - z_1 z_3)\frac{\partial}{\partial z_3} - 2m z_1\right]$$

$$\hat{Q}^2 = -\frac{i}{2}\left[-(z_1^2 + 1)\frac{\partial}{\partial z_1} - z_1 z_2 \frac{\partial}{\partial z_2} - (z_2 - z_1 z_3)\frac{\partial}{\partial z_3} + 2m z_1\right]$$

$$\hat{Q}^3 = z_1 \frac{\partial}{\partial z_1} + \frac{z_2}{2}\frac{\partial}{\partial z_2} - \frac{z_3}{2}\frac{\partial}{\partial z_3} - m$$

$$\hat{Q}^4 = -\frac{1}{2}\left[z_1 z_2 \frac{\partial}{\partial z_1} + (z_2^2 - 1)\frac{\partial}{\partial z_2} + z_3(z_2 - z_1 z_3)\frac{\partial}{\partial z_3} - 2m z_2 - 2n(z_2 - z_1 z_3)\right]$$

$$\hat{Q}^5 = -\frac{i}{2}\left[-z_1 z_2 \frac{\partial}{\partial z_1} - (z_2^2 + 1)\frac{\partial}{\partial z_2} - z_3(z_2 - z_1 z_3)\frac{\partial}{\partial z_3} + 2m z_2 + 2n(z_2 - z_1 z_3)\right] \tag{3.15}$$

$$\hat{Q}^6 = \frac{1}{2}\left[-z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} + (z_3^2 - 1)\frac{\partial}{\partial z_3} - 2n z_3\right]$$

$$\hat{Q}^7 = \frac{i}{2}\left[z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} - (z_3^2 + 1)\frac{\partial}{\partial z_3} + 2n z_3\right]$$

$$\hat{Q}^8 = \frac{\sqrt{3}}{2}(z_2 \frac{\partial}{\partial z_2} + z_3 \frac{\partial}{\partial z_3}) - \frac{1}{\sqrt{3}}m - \frac{2}{\sqrt{3}}n.$$

We find that the expression (3.13) reduces to the above one after naively using the Fock-Bargmann representation $a_i \rightarrow \partial/\partial z_i, a_i^\dagger \rightarrow z_i$ with the substitution $z_4 = z_2 - z_1 z_3$ and acting

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on the physical states annihilated by $a_4, \frac{\partial}{\partial z_4}|\text{phys} >= 0$. However, the relation between the two approach must be investigated further: the Eqs. (3.13) and (3.14) which correspond to the process of reduction after quantization, in general, does not give the same result as the case of quantization after reduction, Eq. (3.15).

IV. CONCLUSION

We studied the HP oscillator realization on the coadjoint orbits of the $SU(N+1)$ and $SU(1, N)$ group by considering the symplectic reduction of these group and by using the action-angle variables transformations. The HP expressions were obtained after canonical quantization with a suitable normal ordering. In the minimal case, the constraints can be solved explicitly but in the maximal case, some of the constraints were imposed directly on the physical states. The corresponding Dyson realizations were also obtained.

It would be straightforward to extend the above formalism to other coadjoint orbits. Especially, it would be interesting to apply it in studying the generalized spin system, ferromagnet or antiferromagnet system on the flag manifold [10] and the Hermitian symmetric space [25,26]. Finally, the q-deformation of the Eqs. (3.11) and (3.13) poses another interesting problem. Details will appear elsewhere.

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*   E-mail address: ploh@dirac.skku.ac.kr

**   E-mail address: rim@phy0.chonbuk.ac.kr

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